ON THE SYMMETRY SOLUTIONS OF TWO-DIMENSIONAL SYSTEMS NOT SOLVABLE BY STANDARD SYMMETRY ANALYSIS

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ABSTRACT. A class of two-dimensional systems of second-order ordinary differential equations is identified in which a system requires fewer Lie point symmetries than required to solve it. The procedure distinguishes among those which are linearizable, complex-linearizable and solvable systems. We also present the underlying concept diagrammatically that provides an analogue in $\Re^3$ of the geometric linearizability criteria in $\Re^2$.

1. INTRODUCTION

One method of solving a nonlinear system of ordinary differential equations (ODEs) is to reduce it to linear form by invertible transformations of independent and dependent variables (point transformations). Lie presented the most general form of a scalar second-order linearizable ODE by considering arbitrary point transformations [1]. Over the past few years there has been a rapidly growing interest in studying linearization problems for higher-order ODEs and systems of second-order ODEs. Linearizability of systems of two second-order quadratically semi-linear (in the first-order derivative) ODEs is addressed in [2, 3] using different algebraic techniques. A geometric approach was used to obtain linearization criteria for systems of two second-order cubically semi-linear ODEs obtainable by projecting down a system of three geodesic equations [4]. Indeed, utilizing arbitrary point transformations the most general form of linearizable systems of two second-order ODEs has been obtained [5]. The use of generalized Sundman transformations in linearization problems is done in [6]. The construction of linearizing transformations from the first integrals of two-dimensional systems is carried out in [7, 8].

Nonlinear systems of two second-order ODEs transformable to linear systems with constant coefficients have three equivalence classes depending on the dimensions of the symmetry algebras (7, 8 or 15) [9]. An optimal canonical form (simplest linear form with variable coefficients) was established to provide dimensions of Lie point symmetry algebras for linearizable systems of two second-order ODEs [10]. This simplest form involves three parameters, whose specific choices gave five equivalence classes for systems of two second-order linearizable ODEs. It was proved that there exist 5, 6, 7, 8 or 15−dimensional Lie algebras for such systems.

Recently we found *special linearizable classes* of those two-dimensional systems [11], namely with 6, 7 and 15−dimensional algebras, that correspond to a complex linearizable scalar second-order ODE [12, 13]. (The characterization for the correspondence [11] will be explained in the next section.) The linearizing transformations to map nonlinear systems to linear form were provided by complex transformations of the form

$$\mathcal{L}_1 : (x, u(x)) \rightarrow (\chi(x), U(x, u)),$$

where $u(x)$, is an analytic complex function. A symmetry solution of a scalar second-order ODE can be obtained if it has a specific (solvable) two dimensional algebra [14], while a system of two second-order ODEs requires at least four symmetry generators to be solvable [15]. We investigate a *special non-linearizable class* of systems of two ODEs which is also obtainable from a linearizable scalar complex equation, but the complex linearizing transformations in this case are different from those given above, i.e.,

$$\mathcal{L}_2 : (x, u(x)) \rightarrow (\chi(x, u), U(x, u)).$$

These transformations cannot be used to obtain the linearizing transformations for the corresponding system. The systems in this class are non-linearizable as they have at most four Lie point symmetries [10]. However, in the case of four symmetry generators they could be solved in principle. For less than four generators *they are not solvable* by usual symmetry analysis. Nevertheless, they may correspond to a scalar linearizable complex ODE and the resulting solution would also solve the original system. We call this procedure for solving systems of two second-order ODEs with less than five symmetries *complex linearization*. In this way we classify two-dimensional systems into three classes. The complex-linearizable systems of dimension d of their symmetry algebras with $d \leq 4$ and $d > 4$, respectively, form two mutually disjoint classes. On the other hand two-dimensional systems can also be solved by projecting them to solvable scalar complex equations regardless of whether they are complex-linearizable or not. We form another class of such systems.

The procedure of constructing a pair of real functions of two variables from a single complex function of a complex variable, leading to a system of PDEs, entails the use of the Cauchy-Riemann equations (CR-equations) in a transparent way. However, the role of these equations for a system of ODEs is far from clear as the CR-equations require two independent variables. Here we give a simple explanation of this role, with a diagrammatic description, for the free particle second-order complex ODE which presents an elegant framework of viewing the straight line (which is maximally symmetric and invariant under the full group $SL(3, \mathbb{R})$) in a higher dimensional space which is only possible if we put on the *complex glasses*.

The outline of the paper is as follows. The criteria for the correspondence of systems with complex linearizable equations are given in the second section. The third section is devoted to the role of the CR-equations associated with systems of two second-order ODEs. The subsequent section contains applications of several coupled systems which are candidate of complex linearization. The last section contains the conclusion and discussion.

2. Systems of ODEs obtainable from complex scalar equations

We first explain the basic formalism of complex linearizability by taking a general system of two second-order ODEs

$$\begin{aligned}
f_1'' &= \omega_1(x, f_1, f_2, f_1', f_2'), \\
f_2'' &= \omega_2(x, f_1, f_2, f_1', f_2'),
\end{aligned} \tag{1}$$

which may be regarded as a surface in a $2(3)+1 = 7-$dimensional space S whose components comprising of independent and dependent variables along with their derivatives. Hence a solution of system (1) is an integral curve on this surface in S. We now introduce a complex structure $J_{\mathbb{C}} : \mathbb{R}^6 \rightarrow \mathbb{C}^3$, on the $6-$dimensional subspace of S by assuming $f(x) + ig(x) = u(x)$, where all first-order and second-order derivatives are determined with u' and u'', respectively. Therefore our solution curve is now embedded in a complex $3_{\mathbb{C}}-$dimensional space, $\mathbb{C}^3$. If we

regard x as the dimension of time then this can be viewed as the propagation of our solution curve in time in a complex space $\mathbb{C}^3$. This identification gives deeper insights into the symmetry analysis which we shall see as we proceed to the subsequent sections. This yields a class of those two-dimensional systems (1) which can be projected to a scalar second-order complex equation

$$u'' = w(x, u, u'), \tag{2}$$

where $w(x) = \omega_1(x) + i\omega_2(x)$, in a $3_{\mathbb{c}} + 1 = 4-$dimensional partially complex space $S_{\mathbb{c}}$, which is comprised of a $3_{\mathbb{c}}-$complex dimensional subspace and a one-dimensional subspace that correspond to independent variable x which is not complex. The basic criterion to identify such systems is that both ω_1 and ω_2 in (1), satisfy the Cauchy Riemann equations

$$\frac{\partial \omega_1}{\partial \bar{u}} = 0, \quad \frac{\partial \omega_1}{\partial \bar{u}'} = 0,$$
$$\frac{\partial \omega_2}{\partial \bar{u}} = 0, \quad \frac{\partial \omega_2}{\partial \bar{u}'} = 0, \tag{3}$$

namely, both functions are analytic. This line of approach has been followed in [11, 12, 13]. Several non-trivial and interesting results were obtained for two-dimensional systems *despite* "trivial" identification of systems (1) from scalar equations (2). Here we derive two main results. Firstly the geometrical picture of complex-linearizability is developed which brings an analogous description for two-dimensional systems to the scalar equation in which the linear target equation $u'' = 0$, represents a straight line. Then we solve those two-dimensional systems which have fewer symmetries 4, 3, 2, 1, i.e., which can not be reduced to quadratures using standard Lie approach.

The use of complex transformations in linearizing a system of two ODEs is studied in [13, 11]. They provided a class of cubically semi-linear two-dimensional systems that can be a candidate of complex linearization. A scalar second-order differential equation is linearizable provided it has the maximal symmetry algebra $sl(3, \mathbb{R})$. A two-dimensional system of ODEs

$$f_1'' = A_1 f_1'^3 - 3A_2 f_1'^2 f_2' - 3A_1 f_1' f_2'^2 + A_2 f_2'^3 + B_1 f_1'^2 - 2B_2 f_1' f_2' -$$
$$B_1 f_2'^2 + C_1 f_1' - C_2 f_2' + D_1,$$
$$f_2'' = A_2 f_1'^3 + 3A_1 f_1'^2 f_2' - 3A_2 f_1' f_2'^2 - A_1 f_2'^3 + B_2 f_1'^2 + 2B_1 f_1' f_2' -$$
$$B_2 f_2'^2 + C_2 f_1' + C_1 f_2' + D_2, \tag{4}$$

where A_i, B_i, C_i, D_i $(i = 1, 2)$ are functions of x and f_i, is a candidate of complex linearization if and only if its coefficients satisfy a set of four constraint equations [13],

$$12A_{1,xx} + 12C_1A_{1,x} - 12C_2A_{2,x} - 6D_1A_{1,f_1} - 6D_1A_{2,f_2} +$$
$$6D_2A_{2,f_1} - 6D_2A_{1,f_2} + 12A_1C_{1,x} - 12A_2C_{2,x} + C_{1,f_1f_1} -$$
$$C_{1,f_2f_2} + 2C_{2,f_1f_2} - 12A_1D_{1,f_1} - 12A_1D_{2,f_2} + 12A_2D_{2,f_1} -$$
$$12A_2D_{1,f_2} + 2B_1C_{1,f_1} + 2B_1C_{2,f_2} - 2B_2C_{2,f_1} + 2B_2C_{1,f_2} -$$
$$8B_1B_{1,x} + 8B_2B_{2,x} - 4B_{1,xf_1} - 4B_{2,xf_2} = 0,$$
$$12A_{2,xx} + 12C_2A_{1,x} + 12C_1A_{2,x} - 6D_2A_{1,f_1} - 6D_2A_{2,f_2} -$$
$$6D_1A_{2,f_1} + 6D_1A_{1,f_2} + 12A_2C_{1,x} + 12A_1C_{2,x} + C_{2,f_1f_1} -$$
$$C_{2,f_2f_2} - 2C_{1,f_1f_2} - 12A_2D_{1,f_1} - 12A_2D_{2,f_2} - 12A_1D_{2,f_1} +$$
$$12A_1D_{1,f_2} + 2B_2C_{1,f_1} + 2B_2C_{2,f_2} + 2B_1C_{2,f_1} - 2B_1C_{1,f_2} -$$
$$8B_2B_{1,x} - 8B_1B_{2,x} - 4B_{2,xf_1} + 4B_{1,xf_2} = 0,$$
$$24D_1A_{1,x} - 24D_2A_{2,x} - 6D_1B_{1,f_1} - 6D_1B_{2,f_2} + 6D_2B_{2,f_1} -$$
$$6D_2B_{1,f_2} + 12A_1D_{1,x} - 12A_2D_{2,x} + 4B_{1,xx} - 4C_{1,xf_1} -$$
$$4C_{2,xf_2} - 6B_1D_{1,f_1} - 6B_1D_{2,f_2} + 6B_2D_{2,f_1} - 6B_2D_{1,f_2} +$$
$$3D_{1,f_1f_1} - 3D_{1,f_2f_2} + 6D_{2,f_1f_2} + 4C_1C_{1,f_1} + 4C_1C_{2,f_2} -$$
$$4C_2C_{2,f_1} + 4C_2C_{1,f_2} - 4C_1B_{1,x} + 4C_2B_{2,x} = 0,$$
$$24D_2A_{1,x} + 24D_1A_{2,x} - 6D_2B_{1,f_1} - 6D_2B_{2,f_2} - 6D_1B_{2,f_1} +$$
$$6D_1B_{1,f_2} + 12A_2D_{1,x} + 12A_1D_{2,x} + 4B_{2,xx} - 4C_{2,xf_1} +$$
$$4C_{1,xf_2} - 6B_2D_{1,f_1} - 6B_2D_{2,f_2} - 6B_1D_{2,f_1} + 6B_1D_{1,f_2} +$$
$$3D_{2,f_1f_1} - 3D_{2,f_2f_2} - 6D_{1,f_1f_2} + 4C_2C_{1,f_1} - 4C_2C_{2,f_2} +$$
$$4C_1C_{2,f_1} - 4C_1C_{1,f_2} - 4C_2B_{1,x} - 4C_1B_{2,x} = 0. \tag{5}$$

The system (4) corresponds to a cubically semilinear complex scalar equation of the form

$$u'' + E_3(x, u)u'^3 + E_2(x, u)u'^2 + E_1(x, u)u' + E_0(x, u) = 0. \tag{6}$$

where E_i, $(i = 0, 1, 2, 3)$ are complex functions of x and u. In [11, 13], examples are given to illustrate the usefulness of this correspondence. They contain a discussion of the optimal canonical form and cases of 6, 7 and 15−dimensional symmetry algebras for two-dimensional linearizable systems. This class of systems was obtained from the scalar linearizable second-order complex equations. There the solvability of systems of the form (4) was achieved by linearizing them. The linearizing transformations were obtained from the complex linearizing transformations $\mathcal{L}_1$, i.e., transformations of the form $(real, complex) \rightarrow (real, complex)$ of the associated complex linearizable equations. The non-linearizable class of systems we are addressing here is associated with those complex nonlinear equations that are transformable to the complex free particle equation via complex transformations of the order $(real, complex) \rightarrow (complex, complex)$, namely

$$(x, u(x)) \rightarrow (\chi(x, u), U(\chi(x, u))).$$

Both the classes (linearizable/non-linearizable) of systems associated with complex scalar linearizable equations are disjoint relative to dimensions of their symmetry algebras. Hence

they are inequivalent under invertible point transformations. For completeness we also examine the solvability of those systems that are not complex-linearizable yet can be solved via complex variables if they can be mapped to solvable scalar complex equations. For this purpose we state the general form of scalar equations with two symmetries. The integration strategies developed to solve a scalar complex second-order ODE (2) require a two parameter group (see, e.g., [14, 15]) called $\mathbf{G}_2$. The integrable forms of a complex second-order equation admitting $\mathbf{G}_2$ are given in the following Table I.

Table I: Lie canonical forms of complex scalar equations

Type	symmetry generators	Representative equations
I	$Z_1 = \partial_x,\ Z_2 = \partial_u,$	$u'' = w(u')$
II	$Z_1 = \partial_u,\ Z_2 = x\partial_u,$	$u'' = w(x)$
III	$Z_1 = \partial_u,\ Z_2 = x\partial_x + u\partial_u,$	$xu'' = w(u')$
IV	$Z_1 = \partial_u,\ Z_2 = u\partial_u,$	$u'' = u'w(x)$

In short we summarize the complex variable approach by presenting the following algorithm.

Algorithm for Solvability: A system of two second-order ODEs (1) is solvable regardless of the number of symmetries if the corresponding complex scalar equation (2) is:
(i) *solvable, i.e., it has a two-parameter group $\mathbf{G}_2$; or*
(ii) *linearizable via invertible complex transformations of the form $\mathcal{L} : (x, u) \to (\chi, U)$.*

3. Classification

We categorize all two-dimensional systems into three classes, of which two are mutually disjoint. We first form a class Υ_1 of those systems that correspond to complex scalar equations of sub algebras of $sl(3, \mathbb{C})$ which are solvable via complex variables whence Υ_1 includes both complex-linearizable and complex-solvable systems which comes from the requirement of two symmetries to reduce a scalar complex equation into quadratures (see Table-1). The most general form of this class is given in (1). We now divide the complex-linearizable systems into two disjoint classes Υ_2 and Υ_3 with respect to their *real* symmetry algebras of dimension $d(\Upsilon_2) > 4$ and $d(\Upsilon_3) \leq 4$, respectively. Since there exists five linearizable classes of two dimensional systems therefore each candidate of $d(\Upsilon_3)$ is non-linearizable in standard Lie theory. The class Υ_3 is the main subject of discussion in this paper. The class Υ_1 includes both complex-linearizable systems as well as non-complex-linearizable systems, i.e.,

$$\Upsilon_1 \cap \Upsilon_2 = \Upsilon_2, \quad \Upsilon_1 \cap \Upsilon_3 = \Upsilon_3. \tag{7}$$

Since [11] contains a discussion on the class Υ_2 therefore we present a short review for completeness. Namely an inhomogeneous geodesic-type two-dimensional system

$$f_1'' + f_1'^2 - f_2'^2 = \Omega_1(x, f_1', f_2'),$$
$$f_2'' + 2f_1'f_2' = \Omega_2(x, f_1', f_2'), \tag{8}$$

where Ω_1 and Ω_2 are linear functions of first derivatives, which have symmetry algebras of dimensions 6, 7 and 15 was found to be complex-linearizable. To reveal integrability the

following real transformations

$$\chi(x) = \frac{1}{x}, \ F_1 = e^{f_1} \cos(f_2), \ F_2 = e^{f_1} \sin(f_2), \tag{9}$$

were used which correspond to the simple complex transformations $\chi = \chi(x)$, $U(\chi) = e^u$, that map the system (8) into a linear system. Note that the non-linearity in (8) comes from the quadratic terms. In order to construct the representative system for the class Υ_3 with fewer symmetries *yet* is complex-linearizable we pick a class of cubically semilinear systems

$$f_1'' = \beta f_1'^3 - 3\gamma f_1'^2 f_2' - 3\beta f_1' f_2'^2 + \gamma f_2'^3,$$
$$f_2'' = \gamma f_1'^3 + 3\beta f_1'^2 f_2' - 3\gamma f_1' f_2'^2 - \beta f_2'^3, \tag{10}$$

where $\beta = \beta(x, f_1, f_2)$ and $\gamma = \gamma(x, f_1, f_2)$. If we assume

$$\beta_{xx} = 0, \quad \gamma_{xx} = 0, \tag{11}$$

i.e., $\beta = b_1 x + b_2$, $\gamma = c_1 x + c_2$, then the coefficients in (10) satisfy (5) therefore the system (10) is complex-linearizable. It is easy to verify that the system (10) with constants b_1, b_2, c_1 and c_2 not zero simultaneously, has symmetry algebra of dimension less than 4, i.e., $d(\Upsilon_3) \leq 4$.

Thus Υ_1, Υ_2 and Υ_3 form three classes of two-dimensional systems. Notice that the class (8) corresponds to the complex scalar equation (6) with $E_3 = 0 = E_0$,

$$u'' + E_2(x, u)u'^2 + E_1(x, u)u' = 0. \tag{12}$$

It includes the cases of those systems which either have $6-$ or $7-$dimensional algebras or correspond to maximally symmetric system which are identified with Lie algebra $sl(15, \mathbb{R})$. Furthermore Υ_2 also contains the reduced optimal canonical form established in [11]. In order to construct Υ_3 we first observe that system (10) is identical to (6)

$$u'' + E_3(x, u)u'^3 = 0, \tag{13}$$

with $E_2 = 0 = E_1 = E_0$. It is in this class that cases of two-dimensional systems of symmetry algebras of dimensions 4, 3, 2 and 1 arise. In the examples of the subsequent section we establish their complex-linearizability besides complete integrability. Since the complex algebra of the systems in the classes Υ_2 and Υ_3 is not equal, therefore there do not exist any complex transformations that map a system in Υ_2 to a system in Υ_3 and vice versa. Hence the complex-linearizable systems (10) and (8) respectively, are mutually disjoint, i.e.,

$$\Upsilon_2 \cap \Upsilon_3 = \emptyset. \tag{14}$$

In short, the class Υ_2 contains those complex-linearizable systems which are at most quadratically semi-linear and they can be mapped to the optimal canonical form [11] whereas all complex-linearizable systems with cubic nonlinearity are contained in Υ_3 and there exists *no* point transformation $T : \Upsilon_2 \to \Upsilon_3$ that maps a system in Υ_2 to one in Υ_3.

4. Role of the CR-equations for systems of ODEs

We know that all linearizable scalar differential equations are equivalent to the free particle equations whose solution is a straight line. The crucial step after ensuring complex linearizability is to obtain the transformations which help in the integration of the systems in Υ_2 and Υ_3. In [11] the complex transformations of the following form

$$\mathcal{L} : (x, u) \to (\chi(x), U(x, u)), \tag{15}$$

given by (9) were used to map a system in class Υ_2 into the free particle complex equation, $U'' = 0$, where prime denotes differentiation with respect to χ. The realification (splitting into real and imaginary parts) of the later gives

$$F_1'' = 0, \quad F_2'' = 0, \tag{16}$$

to which a complex-linearizable system of $6-$ and $7-$dimensional symmetry algebra *can not be* mapped, in general, using classical linearizing transformations as established in [11]. The transformations (9) was obtained from the realification of transformations (15). It is noteworthy that the order of transformations in this case is $(real, complex) \to (real, complex)$. Using complex variable approach a system (1) can be mapped to a simpler system via general transformations of the form

$$\mathcal{L} : (x, u) \to (\chi(x, u), U(x, u)), \tag{17}$$

namely, $(real, complex) \to (complex, complex)$, in which the first argument χ, can be a complex function thereby adding a superficial dimension. In particular, upon realification

$$\chi(x, u) = \chi_1(x, f_1, f_2) + i\chi_2(x, f_1, f_2), \tag{18}$$

and since the dependent function $U(\chi)$ is a complex dependent function which yields two real functions F_1 and F_2, both of them are not only functions of χ_1 but also of χ_2, therefore the linearized scalar equation, $U'' = 0$, fails to produce the free particle system (16). Notwithstanding, the solution of a system in classes Υ_1, Υ_2 and Υ_3 is extractable from the complex solution $u(x)$, upon its realification. This is the basic difference between classical and complex linearization. Lastly we pursue the reason behind the linearized equation $U'' = 0$. Since the prime denotes differentiation with respect to χ which upon using the chain rule yields

$$\frac{\partial}{\partial \chi} = \frac{1}{2}\left(\frac{\partial}{\partial \chi_1} - i\frac{\partial}{\partial \chi_2}\right), \tag{19}$$

and so it is a system of two partial differential equations

$$\begin{aligned}
F_{1_{\chi_1\chi_1}} - F_{1_{\chi_2\chi_2}} + 2F_{2_{\chi_1\chi_2}} &= 0, \\
F_{2_{\chi_1\chi_1}} - F_{2_{\chi_2\chi_2}} - 2F_{1_{\chi_1\chi_2}} &= 0.
\end{aligned} \tag{20}$$

Now by definition a complex Lie point transformation is analytic thus $\mathcal{L}$ is analytic. Since the derivative u' transforms into a complex derivative U' which exists if and only if $U(\chi)$ is complex analytic and is preserved under $\mathcal{L}$ therefore

$$F_{1_{\chi_1}} = F_{2_{\chi_2}}, \quad F_{1_{\chi_2}} = -F_{2_{\chi_1}}, \tag{21}$$

which are the famous CR equations. It is the solution of system (20) along with the condition (21) which upon invoking invertible transformations (17) reveal solutions of the original system. Hence we have established the following result.

Theorem 1. *All complex-linearizable two-dimensional systems in classes Υ_2 and Υ_3 can be transformed into system* (20) & (21) *under the transformation* (17).

We now develop some nontrivial and interesting geometrical aspects of complex-linearization. In order to do that we employ the original idea of Riemann, that a complex function may

be regarded as the dependence of one plane on another plane, unlike the dependence of a real function on a line. To do this we first obtain the solution of system $(20)-(21)$ which is

$$F_1(\chi_1, \chi_2) = c_1\chi_1 + c_2\chi_2 + c_3,$$
$$F_2(\chi_1, \chi_2) = c_1\chi_2 - c_2\chi_1 + c_4, \tag{22}$$

where c_i, $(i = 1, 2, 3, 4)$ are real arbitrary constants. These are two coordinate planes determined by χ_1 and χ_2 with normals

$$\mathbf{n}_1 = [c_1, \ c_2],$$
$$\mathbf{n}_2 = [c_2, -c_1], \tag{23}$$

thus they intersect at right angles

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0, \tag{24}$$

resulting in a straight line at intersection. Thus the geometric linearizing criterion for scalar second-order differential equations, namely a straight line, is extended to the intersection of two planes at right angle in the complex linearization of two-dimensional systems. Note that both χ_1 and χ_2 in (18) are functions of (x, f_1, f_2). Therefore the role of χ_2 can be regarded as slicing the three-dimensional space $\mathbb{R}^3 = \{(x, f_1, f_2)\}$ into two coordinate planes. Interestingly the solution (f_1, f_2) of the system under consideration is found by solving (22) with the use of F_1 and F_2 from $U(\chi)$ in (17). Hence we arrive at the following geometrical result.

Theorem 2. *The necessary and sufficient condition for a two-dimensional system (4) to be complex-linearizable is that the two planes determined by (22) intersect at right angle resulting in a straight line which corresponds to scalar linear equations.*
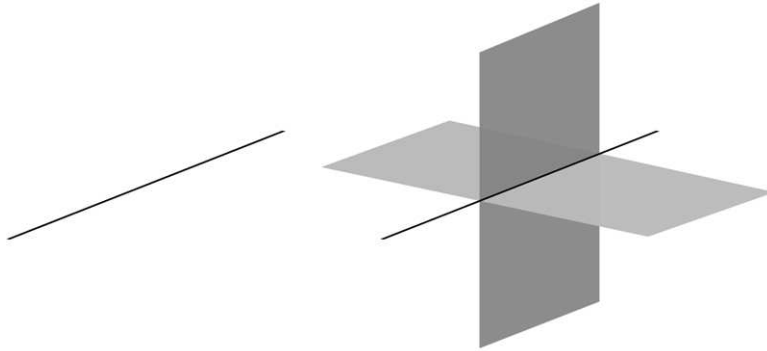

FIGURE 1. Geometry of complex-linearization: The straight line embedded in a three-dimensional space $\mathbb{R}^3$ as the intersection of two mutually perpendicular planes.

Figure 1 illustrates the geometry and presents an elegant description of complex-linearization. In the end it is important to mention a remark that the cases in which the independent variable $\chi(x)$ is *only* a function of x (hence a real function) as in the examples given in [11] on

class Υ_2, the system (8) reduces to the free particle system (16). In these cases the linearizing transformations can be derived after applying the process of realification on the complex transformation.

5. APPLICATIONS

We now illustrate the theory with the aid of examples on Υ_1, Υ_2 and Υ_3. The complex-linearizable systems with algebras of dimensions 4, 3, 2, 1 which is the main topic of discussion are given in the end.

1. Υ_1-**class:** This class contains both complex-linearizable and complex-solvable systems. As we have already seen examples on the former therefore we present an example of the later by integrating a system with two symmetries from complex variable approach. Consider a nonlinear coupled system

$$f_1'' = \frac{(f_1^2 - f_2^2)f_1'}{(f_1^2 - f_2^2)^2 + 4f_1^2 f_2^2} + \frac{2f_1 f_2 f_2'}{(f_1^2 - f_2^2)^2 + 4f_1^2 f_2^2},$$
$$f_2'' = \frac{(f_1^2 - f_2^2)f_2'}{(f_1^2 - f_2^2)^2 + 4f_1^2 f_2^2} - \frac{2f_1 f_2 f_1'}{(f_1^2 - f_2^2)^2 + 4f_1^2 f_2^2}, \tag{25}$$

which has a two$-$dimensional algebra $[X_1, X_2] = 2X_1$, where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x\frac{\partial}{\partial x} + f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}. \tag{26}$$

Using standard Lie analysis it is not straight-forward to carry out integration of this system. Here we highlight the crucial steps involved in using the complex transformations (17) in the form of invariants and differential invariants of symmetries. We first observe that w_1 and w_2 in (1), given by the right hand sides of system (25) satisfy CR-equations (3) therefore system (25) can be mapped to a scalar complex equation. Indeed, the equation

$$u'' = \frac{u'}{u^2}, \tag{27}$$

corresponds to system (25) and it has two complex symmetries X_1 and $2x\partial_x + u\partial_u$, therefore (27) has a solvable Lie algebra $\mathbf{G}_2$. The integration of above equation can be obtained using both approaches, canonical coordinates or differential invariants. Since scaling is inherited under X_1 therefore we employ canonical coordinates relative to symmetry X_1. The canonical transformation

$$\chi = u, \quad \psi = x, \quad U(\chi) = \frac{d\psi}{dx} = \frac{1}{u'},$$

convert (27) into a first-order equation

$$U' = \frac{U^2}{\chi^2}, \tag{28}$$

which upon realification yields a system of partial differential equations as χ is a complex independent variable. That is why the system (25) is reduced to a pair of first-order partial differential equations rather ODEs. This is a similar situation that arises in complex linearization except the difference that here the target equation is a reduced solvable ODE not

a linear equation. By integrating the above equation and using invertible transformation we obtain the solution

$$2c_1 f_1 + \ln\left((c_1 f_1 - 1)^2 + c_2^2 f_2^2\right) - 2c_1^2 x - 2c_1^2 c_2 = 0,$$

$$c_1 f_2 + \arctan\left(\frac{c_1 f_1 - 1}{c_1 f_2}\right) = 0, \tag{29}$$

of system (25).

2. Υ_2−class: For completeness we revisit a case of system which has a $7-$dimensional Lie algebra [11]. It is important to highlight the use of complex transformations at the two steps involved in its complex linearization. In this case, we assume Ω_1 and Ω_2 in (8) as some linear functions of the first derivatives f_1', f_2', i.e.,

$$f_1'' + f_1'^2 - f_2'^2 = c_1 f_1' - c_2 f_2',$$
$$f_2'' + 2f_1' f_2' = c_2 f_1' + c_1 f_2', \tag{30}$$

which admits 7-dimensional algebra, provided both c_1 and c_2, are not simultaneously zero. Using the transformation

$$\chi(x) = x, \ \ F_1 = e^{f_1} \cos(f_2), \ \ F_2 = e^{f_1} \sin(f_2), \tag{31}$$

relative to (9), it can be mapped to a linear system

$$F_1'' = c_1 F_1' - c_2 F_2',$$
$$F_2'' = c_2 F_1' + c_1 F_2', \tag{32}$$

which also has a 7-dimensional symmetry algebra. Since, the number of symmetries are same for both systems it ensures that there exists some linearizing point transformation. To find it we put on *complex glasses* and observe that the corresponding complex equation $u'' + u'^2 = cu'$, can be mapped invertibly to $U'' = cU'$, using (9), which is linear, all linear scalar equations are equivalent it can be transformed into a complexified free particle equation $\widetilde{U}'' = 0$, using the simple mapping

$$(\chi(x), U) \ \rightarrow \ (\widetilde{\chi} = \alpha + \beta e^{c\chi(x)}, \widetilde{U} = U), \tag{33}$$

where α and β are complex constants. It is vital to identify the distinction between the two crucial steps in the above analysis which we elaborate here. In the first step, we use the process of realification of complex variables (9) to find the real transformation that maps system (32) directly. It is in the second step, namely $(\chi, U) \rightarrow (\widetilde{\chi}, \widetilde{U})$, that we can not apply realification directly to extract transformation that maps our system to maximally symmetric system in general. Because, the independent variable $\widetilde{\chi}$ is complex so long as c is a complex number, i.e., $c = c_1 + ic_2$, it jumps off the real line by adding an extra dimension superficially and results in a set of linear partial differential equations. Elegantly, this larger $4-$dimensional space $(\chi_1, \chi_2, F_1(\chi_1, \chi_2), F_2(\chi_1, \chi_2))$, where χ_2 adds the superficial dimension, is also equipped with an analytic structure contains the general solution curve of our system (32). It is noteworthy an *independent* dimension due to complex c, is

$$e^{c\chi(x)} = e^{c_1 \chi} \cos(c_2 \chi) + ie^{c_1 \chi} \sin(c_2 \chi). \tag{34}$$

Hence, $\chi_2(x) = e^{c_1 \chi} \sin(c_2 \chi(x))$. Thus, the solution of system (32) can be found from the complex solution of the linearized equation by applying the inverse transformations. Clearly,

this hidden connection can only be uncovered if we work in the field of complex variables right from the start.

3. Υ_3-**class:** The subsequent examples briefly explain complex-linearizability for systems with fewer symmetries. It is due to the elegance of complex variables that systems which can not be dealt via standard Lie symmetry approach are solvable via symmetry approach!

a. Solvable system of $4-$**dimensional algebra:** Considering $\beta(x, f_1, f_2) = 1$ and $\gamma(x, f_1, f_2) = 0$, in (10) we obtain a coupled system

$$
\begin{aligned}
f_1'' - f_1'^3 + 3f_1'f_2'^2 &= 0, \\
f_2'' - 3f_1'^2 f_2' + f_2'^3 &= 0,
\end{aligned}
\tag{35}
$$

which is complex-linearizable and has only four symmetries

$$
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial f_1}, \quad X_3 = \frac{\partial}{\partial f_2}, \quad X_4 = 2x\frac{\partial}{\partial x} + f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2},
\tag{36}
$$

with Lie algebra

$$
\begin{aligned}
[X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = 0, \\
[X_1, X_4] = 2X_1, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = X_3,
\end{aligned}
\tag{37}
$$

therefore, it is not in one of the linearizable classes of two-dimensional systems. Now in order to carry out integration of system (35), we linearize the corresponding equation

$$
u'' - u'^3 = 0.
\tag{38}
$$

which has an $8-$dimensional Lie algebra. This can be mapped to the linear equation

$$
U'' + 1 = 0,
\tag{39}
$$

by inverting the role of the independent and dependent variables $\chi = u$, $U = x$. It has the solution $2U = -\chi^2 + a\chi + b$, where a and b are complex constants, which in terms of the original variables becomes $u(x) = \pm\sqrt{a - 2x} + b$, and yields the solution

$$
f_1(x) = \pm\left(\frac{a_1 - 2x + \sqrt{(a_1 - 2x)^2 + a_2^2}}{2}\right)^{1/2} + b_1,
$$

$$
f_2(x) = \pm\left(\frac{-a_1 + 2x + \sqrt{(a_1 - 2x)^2 + a_2^2}}{2}\right)^{1/2} + b_2,
\tag{40}
$$

of (35). System (35) could be solvable by real symmetry analysis because it has four symmetry generators. However, we now proceed to systems that are *not* solvable by real symmetry methods as they have *fewer symmetry generators than four.*

b. Solvable system of $3-$**dimensional algebra:** It is easy to construct a system from (10) which has only three symmetries. For example we observe that in (10) the functions

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β and γ can be at most linear functions of independent variable x. Hence we obtain a complex-linearizable system

$$f_1'' - xf_1'^3 + 3xf_1'f_2'^2 = 0,$$
$$f_2'' - 3xf_1'^2 f_2' + xf_2'^3 = 0,$$

(41)

by involving x linearly in the coefficients to remove the $x-$translation. Thus we obtain the following $3-$dimensional Abelian Lie algebra

$$X_1 = x\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial f_1}, \quad X_3 = \frac{\partial}{\partial f_2}.$$

(42)

We follow the same procedure as developed in the previous case and solve the corresponding nonlinear equation

$$u'' - xu'^3 = 0.$$

(43)

It is transformable to a linear form $U'' = -U(x)$, which gives the solution

$$u(x) = \arctan\left(\frac{x}{\sqrt{a - x^2}}\right) + b.$$

(44)

System (41) is *not* solvable by real symmetry methods but by complex linearization.

c. Solvable system of $2-$dimensional algebras: Consider the system

$$f_1'' - f_1 f_1'^3 + 3f_2 f_1'^2 f_2' + 3f_1 f_1' f_2'^2 - f_2 f_2'^3 = 0,$$
$$f_2'' - f_2 f_1'^3 - 3f_1 f_1'^2 f_2' + 3f_2 f_1' f_2'^2 + f_1 f_2'^3 = 0,$$

(45)

which has only two Lie symmetries

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 3x\frac{\partial}{\partial x} + f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}.$$

(46)

The system (45) is solvable due to its correspondence with the complex scalar second-order ODE

$$u'' - uu'^3 = 0,$$

(47)

which *is* linearizable, despite having only a two-dimensional algebra. Notice that even a scalar second-order ODE requires at least two symmetries to be solvable, while here we have solved a system of two ODEs. In the subsequent section we provide the solution for a system with *only one* symmetry generator, *which is insufficient to solve even a scalar second-order ODE*. Nevertheless, *we can go further* !

d.   Solvable system of $1-$dimensional algebra: Consider $\beta(x, f_1, f_2) = xf_1$ and $\gamma(x, f_1, f_2) = xf_2$, in (10) we obtain

$$f_1'' - xf_1 f_1'^3 + 3xf_2 f_1'^2 f_2' + 3xf_1 f_1' f_2'^2 - xf_2 f_2'^3 = 0,$$
$$f_2'' - xf_2 f_1'^3 - 3xf_1 f_1'^2 f_2' + 3xf_2 f_1' f_2'^2 + xf_1 f_2'^3 = 0.$$

(48)

This system is non-linearizable as it has *only* a scaling symmetry $X_1 = x\partial_x$. The corresponding scalar second-order complex ODE is

$$u'' - xuu'^3 = 0,$$

(49)

which has an 8–dimensional algebra and linearizes to

$$U'' + \chi U = 0, \tag{50}$$

which is the Airy equation whose solutions are Airy functions extended to the complex plane. The solution of the complex linearized equation for $U(x)$ is given by

$$U(\chi) = c_1 \mathrm{Ai}(-\chi) + c_2 \mathrm{Bi}(-\chi), \tag{51}$$

where $\mathrm{Ai}(-\chi)$ and $\mathrm{Bi}(-\chi)$, are the two Airy functions. Inverting (51), we obtain a solution of the associated nonlinear equation which implicitly provides a solution

$$\begin{aligned}
\Re(c_1 \mathrm{Ai}(-f_1 - if_2) + c_2 \mathrm{Bi}(-f_1 - if_2)) &= x, \\
\Im(c_1 \mathrm{Ai}(-f_1 - if_2) + c_2 \mathrm{Bi}(-f_1 - if_2)) &= 0,
\end{aligned} \tag{52}$$

where $\Re$ and $\Im$ are the real and imaginary parts of the arguments, for the system (48).

e. Coupled-Modified-Emden System (Revisited): Now we revisit an example of a physical system described in [13], consider

$$\begin{aligned}
f_1'' &= -3f_1 f_1' + 3f_2 f_2' - f_1^3 + 3f_1 f_2^2, \\
f_2'' &= -3f_2 f_1' - 3f_1 f_2' + f_2^3 - 3f_1^2 f_2.
\end{aligned} \tag{53}$$

This system has three symmetries X_1, X_2, X_3, where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - f_1\frac{\partial}{\partial f_1} - f_2\frac{\partial}{\partial f_2},$$

$$X_3 = x^2\frac{\partial}{\partial x} - 2xf_1\frac{\partial}{\partial f_1} - 2xf_2\frac{\partial}{\partial f_2}, \tag{54}$$

with Lie algebra $[X_1, X_2] = X_1$, $[X_1, X_3] = 2X_2$ and $[X_2, X_3] = X_3$. The system (53) is solvable *only* by complex linearization. The complex magic played by complex transformations can be seen when we use

$$\chi = x - \frac{1}{u}, \quad U = \frac{x^2}{2} - \frac{x}{u}, \tag{55}$$

to map the nonlinear complex equation into the complex free particle equation, whose solution after inverting the above transformations directly yields the solution of the system (53) [13]. In this case (20) and (21) yield

$$\begin{aligned}
F_1 &= a_1\chi_1 - a_2\chi_2 + b_1, \\
F_2 &= a_2\chi_1 + a_1\chi_2 + b_2,
\end{aligned} \tag{56}$$

where

$$\chi_1 = x - \frac{f_1}{f_1^2 + f_2^2}, \quad \chi_2 = \frac{f_1}{f_1^2 + f_2^2}. \tag{57}$$

From (55), we obtain

$$F_1 = \frac{x^2}{2} - \frac{f_1}{f_1^2 + f_2^2}, \quad F_2 = \frac{f_2}{f_1^2 + f_2^2}. \tag{58}$$

13

Now by solving (56), for f_1 and f_2, by invoking equations (57) and (58), we get the same solution

$$f_1(x) = \frac{2x^3 - 6x^2 a_1 + 4(a_2^2 + a_1^2 - b_1)x + 4a_1 b_1 + 4a_2 b_2}{x^4 - 4x^3 a_1 + 4((a_2^2 + a_1^2 - b_1)x^2 + 2(a_2 b_2 + a_1 b_1)x + b_1^2 + b_2^2)},$$

$$f_2(x) = \frac{(2x^2 + 4b_1)a_2 + 4b_2(x - a_1)}{x^4 - 4x^3 a_1 + 4((a_2^2 + a_1^2 - b_1)x^2 + 2(a_2 b_2 + a_1 b_1)x + b_1^2 + b_2^2)}, \tag{59}$$

of system (53), as obtained in [13] up to a redefinition of constants.

6. CONCLUSION

Real symmetry analysis requires at least a 2 and a 4−dimensional Lie point symmetry algebra to solve a scalar equation and a system of two second-order equations respectively. The scalar nonlinear equation can be mapped to linear form if and only if it has an 8−dimensional algebra. For systems we have five equivalence classes of linearizable systems having 5, 6, 7, 8 or 15−dimensional algebra.

A straight line representing the solution of the (complex) free particle equation must be the intersection of two planes representing the complex linearizable system of ODEs, at right angles to incorporate the CR-equations. Thus we see that complex extensions are very important and they generalize our understanding in a natural way.

Complex symmetry analysis provides a class of systems of two ODEs from a scalar second-order equation if the dependent variable is a complex function of a real independent variable. The linearizability of this base complex equation generates two different classes of systems of two second-order ODEs; (a) real linearizable systems; and (b) complex linearizable systems. Real linearizable systems are proved to have 6, 7 or 15−dimensional algebra [11]. The second class is investigated here and it is found that these systems are not linearizable but are solvable due to their correspondence with linearizable scalar complex equations. This class contains solvable systems with 1, 2, 3 and 4−dimensional algebras. Clearly, from the symmetry algebra the linearizability of such systems is not achievable by real symmetry analysis. Indeed, systems cannot be solved with less than four symmetry generators.

A class of systems can also be generated using CSA where the systems correspond to a non-linearizable but solvable complex scalar equation. While all the contributions of the eight complex symmetries in two-dimensional systems must be comprehended. Namely these complex symmetries yield at least 14 operators by splitting the complex dependent variable, all of them are not symmetries of the system, but they play a crucial role. By assuming a complex variational structure on complex scalar equations, a complex Lagrangian can be used to yield nontrivial first integrals of two-dimensional systems. This requires a classification of Noether symmetries and first integrals of such systems.

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